PHYS 798C Spring 2024
Lecture 3 Summary

Prof. Steven Anlage

I. THE TWO FLUID MODEL OF SUPERCONDUCTORS

We want to understand the screening and low-dissipation properties of superconductors. As a first cut, we introduced the empirical "two fluid" model of superconducting electrodynamics. The model is very simple, but contains a number of results that are consistent with experiment, and qualitatively in agreement with full microscopic theory. A superconductor has two independent fluids, one made up of superconducting electrons, the other of normal electrons, and these fluids inter-penetrate and act in parallel, but do not interact with each other. The relative abundance of these two fluids changes as a function of temperature. We say that the superfluid has a number per unit volume of $n_s(T)$, while the normal fluid has a number density of $n_n(T)$. The total number density is equal to that of the metal in the normal state: $n_s(T) + n_n(T) = n$, a number density fixed by the nature of the metal.

A. Two-Fluid Complex Conductivity

The particles in each fluid obey a Drude-like equation (Newton's second law of motion):

$$\frac{d(m\vec{v})}{dt} = e\vec{E} - \frac{m\vec{v}}{\tau}$$

Again this ignores the Fermi-Dirac character of electrons in real metals, assumes local electrodynamics, and assumes a single momentum relaxation time scale τ. By applying this equation to the super- and normal-fluids separately, we can define their complex conductivities (through $J_s = \sigma_s E$ and $J_n = \sigma_n E$) as:

$$\sigma_s = \frac{n_s e^2 \tau_s/m}{1 + i\omega\tau_s} \text{ and } \sigma_n = \frac{n_n e^2 \tau_n/m}{1 + i\omega\tau_n}$$

If we take the limit $\tau_s \to \infty$ the superfluid conductivity goes over to $\sigma_s = \sigma_{s1} - i\sigma_{s2} = \frac{\pi}{2}\frac{n_s e^2}{m}\delta(\omega) - i\frac{n_s e^2}{m\omega}$, and the normal fluid becomes $\sigma_n = \frac{n_n e^2 \tau_n}{m}$ with the assumption that we operate at frequencies such that $\omega\tau_n \ll 1$. In fact, this limit is often valid because the energy gap of the superconductor imposes an upper limit on the approximate validity of the two-fluid model, and the gap frequency is on the THz scale. Also, typically the momentum relaxation time is on the scale of pico-seconds, $\tau_n \sim 10^{-12}s$, on the same order as the inverse of the gap frequency. (These arguments are basically valid for elemental superconductors, but may break down for other types of superconductors.)

Note that the weight of the delta-function in σ_s can be derived from the Ferrell-Glover-Tinkham optical conductivity sum rule: $I = \int_0^\infty \sigma_s(\omega)\, d\omega = \frac{\pi}{2}\frac{n_s e^2}{m}$. This integral is directly proportional to the "spectral weight" of the superconductor $\frac{n_s}{m}$. Beware that some authors define the integral over frequency from $-\infty$ to $+\infty$, resulting in a pre-factor of π rather than $\pi/2$.

The total conductivity of the superconductor is the sum of the superfluid and normal fluid pieces: $\sigma = \sigma_s + \sigma_n$. There is a simple circuit analogy that captures this complex conductivity. The superconductor acts as if it is a parallel connection of a resistor R (representing the normal channel) and a pure lossless inductor L_s (representing the superfluid channel). At zero frequency all of the current goes through the inductor, and there is no loss (infinite conductivity). At finite frequency the inductive channel now presents some non-zero impedance ($Z_{super} = i\omega L_s$) and as a result some of the current is shunted into the resistive channel. The relative population of the normal and super channels depends on frequency and temperature as $J_s/J_n = \sigma_{2s}/\sigma_{1n} = \frac{n_s}{n_n}\frac{1}{\omega\tau_n}$. Since $\omega\tau_n \ll 1$ this ratio is usually much larger than 1, meaning that most of the current flows through the super-channel until one reaches frequencies near the superconducting gap frequency, or near the transition temperature where $n_s(T)$ is very small.

The finite resistance that superconductors present to alternating fields can be understood as follows. When an ac electric field is present tangent to the surface of a superconductor, the superfluid is accelerated (London's first equation, $\partial \vec{J}_s/\partial t \sim \vec{E}$) and responds to screen the fields out of the bulk of the superconductor. However, the superfluid has a finite inertia (symbolized by its inductance) and therefore does not respond instantaneously to the time dependent electric field. As a result, some of the co-existing normal fluid (symbolized by the resistor) is exposed to the electric field and produces Ohmic dissipation. The losses grow as a strong power of the ac frequency, as discussed below.

B. Two-Fluid Temperature Dependence

One can introduce the following empirical temperature dependencies for the super- and normal-fluids: $n_s(T) = n[1 - t^4]$ and $n_n(T) = nt^4$, where n is the total electron density and $t = T/T_c$ is the "reduced temperature". With this temperature dependence we can now examine the temperature-dependent magnetic screening length in a superconductor: $\lambda(T) = \sqrt{\frac{m}{\mu_0 n_s(T) e^2}} = \frac{\lambda(0)}{\sqrt{1-t^4}}$, where $\lambda(0) = \sqrt{\frac{m}{\mu_0 n_s(0) e^2}}$. This expression shows that the screening length diverges as $t \to 1$, or in other words as $T \to T_c$, meaning that the Meissner effect gracefully goes away as the material makes the transition in to the normal state.

C. Two-Fluid Frequency Dependence

It is interesting to understand the frequency dependence of the dissipated power in a superconductor. The dissipated power per unit volume can be calculated from $P = Re[\rho]J^2 = Re[1/\sigma]J^2$. This results in $P = \frac{\sigma_1}{\sigma_1^2+\sigma_2^2}J^2$. For a superconductor at "low frequencies" such that $\omega\tau_n \ll 1$, we can take $P \approx \frac{\sigma_1}{\sigma_2^2}J^2$. To good approximation we can take σ_1 to be independent of frequency and we know that $\sigma_2 \propto 1/\omega$ from above, hence for a superconductor we expect $P \propto \omega^2$. The corresponding calculation for a normal metal results in a dissipated power per unit volume $P_n \propto \omega^0$, but the total dissipated power scales as $P_n \propto \omega^{1/2}$ because the normal metal skin depth scales as $\delta \propto 1/\omega^{1/2}$. Note that the screening length in a superconductor is frequency independent to good approximation, making it very useful for transporting high-bandwidth electrical impulses with minimal dispersion. The class web site has a comparison plot of dissipated power vs. frequency for superconductors and normal conductors.

Note that the dissipated power in a superconductor is proportional to the real part of the conductivity: $P_s \propto \frac{\sigma_1}{\sigma_2^2}$. On the other hand, the dissipated power in a normal metal is inversely proportional to its conductivity: $P_n \propto \frac{1}{\sigma_n}$. Hence to minimize loss in a superconductor you actually want to minimize the real part of of its conductivity, in stark contrast with normal metals. This means that you want the un-paired electrons in the superconductor to have a high momentum relaxation rate $1/\tau$. This is another example of the motto, 'bad metals make good superconductors'.

Superconductors are also useful for their low dissipation properties. They are used to construct microwave resonators with high quality factors (Q) for use in superconducting qubits and particle accelerators. Niobium radio frequency accelerator cavities with Q values over 10^{11} are now routinely fabricated in labs around the world.

II. PIPPARD'S COHERENCE LENGTH

Pippard deduced the existence of another length scale in superconductors from measurements of the magnetic penetration depth λ as a function of impurity content in the superconductor. He used pure Sn as the starting material and then added various amounts of In to make alloys containing up to 3% In in solid solution. Pippard found that the zero temperature penetration depth was about 60 nm for pure Sn, but increased systematically as In was added, going up to 100 nm at 3% In. At the same time he found that the T_c and H_c of these alloyed superconductors were the same as for pure Sn, hence the thermodynamic properties were essentially unchanged.

The London penetration depth, which we take to be a temperature independent quantity based on the total electron density (n) of the metal is $\lambda_L = \sqrt{\frac{m}{\mu_0 n e^2}}$. For Sn it has a value of 53 nm. But the London equations, based on $\vec{J}_s = -\frac{1}{\Lambda}\vec{A}$ has no dependence on 'dirt'. On the other hand, Ohm's law is clearly dependent on the mean free path of the normal electrons: $\vec{J}_n = \frac{ne^2 \ell_{mfp}}{mv_F}\vec{E}$. Pippard proposed that the finite mean free path of the normal fluid electrons reduces the effectiveness of the superfluid response in screening. He proposed $\vec{J}_s = -\frac{\xi(\ell_{mfp})}{\xi_0}\frac{1}{\Lambda}\vec{A}$, where ξ_0 is a constant length and $\xi(\ell_{mfp})$ is a length that depends on the mean free path as,

$$\frac{1}{\xi(\ell_{mfp})} = \frac{1}{\xi_0} + \frac{1}{\ell_{mfp}}$$

Note that as a result, $\xi(\ell_{mfp}) \leq \xi_0$. Hence the factor in the modified London equation reduces the superfluid response (J_s) for a given perturbation (A). This in turn enhances the penetration of magnetic field into the superconductor. Define a 're-normalized' London constant,

$$\Lambda' := \Lambda \frac{\xi_0}{\xi(\ell_{mfp})} = \mu_0 \lambda_L^2 \frac{\xi_0}{\xi(\ell_{mfp})} := \mu_0 \lambda^2$$

and we now call λ the magnetic penetration depth. It is related to the London penetration depth as

$$\lambda = \lambda_L \sqrt{\frac{\xi_0}{\xi(\ell_{mfp})}} = \lambda_L \sqrt{1 + \xi_0/\ell_{mfp}}$$

This equation explains the dependence of Pippard's measured penetration depth on the mean free path of quasiparticles in the Sn/In alloys. The equation suggests that disordered superconductors will show an enhanced penetration depth (suppressed screening) without showing a decrease in T_c or H_c. In general, one will measure an enhanced screening length unless one has $\ell_{mfp} \gg \xi_0$. Notice that in the 'dirty limit', meaning $\ell_{mfp} \ll \xi_0$, the screening length is given by $\lambda_{dirty} = \sqrt{\frac{\lambda_L^2 \xi_0}{\ell_{mfp}}}$, which depends on three microscopic length scales!

A. The Coherence Length

It turns out that Pippard had discovered the existence of the 'coherence length', a fundamental length scale later introduced formally by BCS theory. Pippard deduced that this length scale is related to the smallest possible size of a 'superconducting wavepacket'. Here is the argument. Only electrons within $k_B T_c$ of the Fermi energy can play a role in the superconducting response that sets in at T_c (BCS shows that <u>all</u> electrons play a role in superconductivity). The momentum range of these electrons is $\Delta p = \frac{k_B T_c}{v_F}$. Hence by the Heisenberg uncertainty principle, the size scale is $\Delta x \geq \frac{\hbar}{\Delta p} = \frac{\hbar v_F}{k_B T_c}$. He then wrote down a more exact relation for the coherence length: $\xi_0 = a\frac{\hbar v_F}{k_B T_c}$, and deduced that $a = 0.15$ from his screening length data on Sn/In alloys. Years later, BCS showed that this expression is correct, and found $a = 0.18$. In Ginzburg-Landau theory, the coherence length is a measure of how quickly the magnitude of the superconducting order parameter can vary in space. In BCS theory ξ_0 can be roughly interpreted as the size of a Cooper pair in real-space.

Note that high carrier density superconductors with low T_c values, like Al and Sn, have long coherence lengths. In this case $\xi_0 \gg \lambda_L$, and these are known as extreme type-I superconductors. We shall see later that superconductors in this limit have a large and positive energy cost for creating a superconductor/normal boundary in their bulk. Hence superconductivity in these materials tends to be destroyed suddenly and completely at the thermodynamic critical field H_c.